

TOWARDS QUANTUM NONCOMMUTATIVE κ -DEFORMED FIELD THEORY¹

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Abstract

We introduce new κ -star product describing the multiplication of quantized κ -deformed free fields. The κ -deformation of local free quantum fields originates from two sources: non-commutativity of space-time and the κ -deformation of field oscillators algebra - we relate these two deformations. We demonstrate that for suitable choice of κ -deformed field oscillators algebra the κ -deformed version of microcausality condition is satisfied, and it leads to the deformation of the Pauli-Jordan commutation function defined by the κ -deformed mass shell. We show by constructing the κ -deformed Fock space that the use of κ -deformed oscillator algebra permits to preserve the bosonic statistics of n -particle states. The proposed star product is extended to the product of n fields, which for $n = 4$ defines the interaction vertex in perturbative description of noncommutative quantum $\lambda\varphi^4$ field theory. It appears that the classical fourmomentum conservation law is satisfied at the interaction vertices.

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1 Introduction.

Noncommutative space-time geometry was firstly physically motivated by combining together the postulates of general relativity and quantum mechanics [1], [2]; alternatively the noncommutative coordinates were derived in brane-world scenario as describing the space-time manifolds attached to the ends of open strings [3], [4]. As next step it is important to construct in noncommutative space-times the classical and quantum dynamical models, in particular, the noncommutative counterparts of quantum local fields and their perturbative expansions.

In noncommutative deformed QFT there are two possible sources of modifications of the standard formulae for quantum fields:

i) We replace the classical Minkowski space-time describing in standard theory the local field arguments by the noncommutative space-time coordinates (see e.g. [5], [6])

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa^2} \theta_{\mu\nu} (\kappa x_\mu) = \frac{i}{\kappa^2} \theta_{\mu\nu}^{(0)} + \frac{i}{\kappa} \theta_{\mu\nu}^{(1)\rho} x_\rho + \dots, \quad (1)$$

ii) Second change comes from the introduction of modified field quantization rules which replace the known infinite set of canonical creation and annihilation operators. In such a way we modify standard boson/fermion oscillators into the κ -deformed ones.

Recently, there was studied the structure of free quantized noncommutative fields for the "canonical" deformation $\theta_{\mu\nu} \equiv \theta_{\mu\nu}^{(0)}$. Important step in understanding of such a theory was the discovery of Hopf-algebraic twisted quantum symmetry leaving invariant the noncommutativity relation (1) with $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$ [7]-[9]. It was shown in such a case (see [10]-[14]) that the $\theta_{\mu\nu}^{(0)}$ -deformation of quantum free field implies the modification of the field oscillators algebra. It appears that perturbative Gell-Mann-Low expansion for interacting $\theta_{\mu\nu}$ -deformed quantum $\lambda\phi^4$ theory can be identified with the analogous undeformed expansion, obtained by putting $\theta_{\mu\nu} = 0$ [10], [13], [14].

In this paper we shall consider QFT on noncommutative Lie-algebraic space-time (see (1) with $\theta_{\mu\nu}^{(1)\rho} \neq 0$), in particular on κ -deformed Minkowski space [15]-[17]

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0. \quad (2)$$

Our main aim is to formulate the κ -deformation of the theory of free quantum fields in a way implying the κ -deformed version of microcausality condition and to provide a ground for the application to interacting case. The perturbative formulation of interacting κ -deformed QFT as well as deformed Feynman diagrams has been considered previously (see e.g. [18]-[21]), however without considering the κ -deformation of the field oscillators. In particular, two problems important for the construction of physically plausible set of κ -deformed Feynman diagrams were not satisfactorily solved:

α) How to introduce the κ -star multiplication \star_κ of the free quantum fields which leads to κ -version of the microcausality relation². For κ -deformed free fields which can be decomposed into positive and negative frequency parts $\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x)$ we should get ($[A, B]_{\star_\kappa} := A \star_\kappa B - B \star_\kappa A$)

$$[\varphi^{(\pm)}(x), \varphi^{(\pm)}(y)]_{\star_\kappa} = 0, \quad (3)$$

β) How to define the local κ -deformed field vertices which provide the classical Abelian four-momentum conservation law for ingoing and outgoing particles. In such a way physically difficult to

²The standard version of causality condition in κ -deformed theory has to be modified, because the notion of standard light cone is modified under κ -deformation.

accept non-Abelian quantum addition law of the fourmomenta will not appear in the description of the particle scatterings.

An important step which permitted us to solve the problems $\alpha)$ and $\beta)$ was the construction of κ -deformed algebra of creation and annihilation operators proposed in [22]. It is easy to see that the κ -deformed addition law for the fourmomenta due to the nonsymmetric fourmomentum coproduct requires the modification of standard oscillators algebra. It follows (see also [22]) that after the exchange of two κ -deformed particles their three-momentum dependence is changed by the multiplicative factors depending on the energy of the other particle. We shall show that the κ -deformed algebra of oscillators for relativistic free fields in comparison with the case considered in detail in [22] should be suitably generalized by the choice of numerical normalization factors. We shall also argue that if we use in particular way the κ -deformed oscillators algebra one can create from the vacuum the multiparticle states with classical fourmomentum addition law, and obtain the n -particle states obeying standard bosonic symmetry properties.

The plan of our paper is the following. In Sect. 2 we shall describe the known Hopf-algebraic framework of κ -deformed relativistic symmetries which shall be used in our considerations. For the definition of κ -deformed noncommutative free field we shall employ the "symmetric" choice of the noncommutative Fourier transform ([23], [24]). In Sect. 3 we shall introduce new κ -star product describing the multiplication of free quantized κ -deformed local fields. We stress here that our new \star -multiplication is a novelty: besides representing the multiplication of noncommutative κ -Minkowski coordinates it introduces additional κ -dependent change of the mass-shell conditions in the multiplied fields. We shall show that with such a new multiplication \star_κ the commutator is a c -number and one obtains for free quantum κ -deformed K-G fields the following commutation relations

$$[\varphi(x), \varphi(y)]_{\star_\kappa} = \frac{1}{i} \Delta_\kappa(x - y; M^2), \quad (4)$$

where the κ -deformed Pauli-Jordan function $\Delta_\kappa(x; m^2)$ is described by the κ -deformed mass-shell condition

$$\Delta_\kappa(x; M^2) = \frac{i}{(2\pi)^3} \int d^4p \epsilon(p_0) \delta((2\kappa \sinh(p_0/2\kappa))^2 - \vec{p}^2 - M^2) e^{ip_\mu x^\mu}. \quad (5)$$

The relation (4) completes the relations (3) and provide the explicit example of quantum field operator satisfying the κ -causality condition. Leaving more detailed discussion of the κ -causality to our future investigations we recall that the commutator (4) was firstly proposed in [25]. In Sect. 4 we shall introduce new multiplication operation in the algebra of field oscillators and shall describe the full algebra of κ -deformed creation and annihilation operators. In Sect. 5 we shall introduce the corresponding κ -deformed Fock space with n -particle particle states carrying fourmomenta which add accordingly to classical Abelian addition law. Further, in Sect. 6 we consider the $D = 4$ local κ -deformed vertex

$$\lambda \varphi^4(x) \rightarrow \lambda \varphi(x) \star_\kappa \varphi(x) \star_\kappa \varphi(x) \star_\kappa \varphi(x), \quad (6)$$

which in momentum space is proportional to the Dirac delta describing classical conservation law of fourmomenta. Finally, in Sect. 7 we provide final remarks.

2 κ -deformed Poincare symmetries and noncommutative Fourier transforms.

The Hopf-algebraic description of κ -deformed Poincare symmetries with mass-like fundamental deformation parameter κ was introduced in [25], [26], where the standard basis with modified boost sector of Lorentz algebra was proposed. By nonlinear transformation of boost generators one arrives at the bicrossproduct basis [16] with classical Lorentz algebra sector. Here we shall consider a modified bicrossproduct basis, which was discussed earlier in [23], [24], [27] with standard κ -deformed mass-shell condition (see [25]) and classical formulae for the coinverses. Introducing the Poincare algebra generators $M_{\mu\nu} = (M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, N_i = M_{i0})$ and $P_\mu = (P_i, P_0)$ we get the following Hopf algebra relations

a) algebraic sector

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i(\eta_{\mu\sigma}M_{\nu\lambda} - \eta_{\nu\sigma}M_{\mu\lambda} + \eta_{\nu\lambda}M_{\mu\sigma} - \eta_{\mu\lambda}M_{\nu\sigma}) , \quad (7)$$

$$[M_i, P_j] = i\epsilon_{ijk}P_k , \quad (8)$$

$$[N_i, P_j] = i\delta_{ij}e^{\frac{P_0}{2\kappa}} \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} e^{-\frac{P_0}{\kappa}} \vec{P}^2 \right] - \frac{i}{2\kappa} e^{-\frac{P_0}{2\kappa}} P_i P_j , \quad (9)$$

$$[M_i, P_0] = 0 , \quad [N_i, P_0] = i e^{-\frac{P_0}{2\kappa}} P_i , \quad (10)$$

$$[P_\mu, P_\nu] = 0 , \quad (11)$$

b) coalgebraic sector

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i , \quad (12)$$

$$\Delta(N_i) = N_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} e^{-\frac{P_0}{2\kappa}} P_j \otimes M_k , \quad (13)$$

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad (14)$$

$$\Delta(P_i) = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i , \quad (15)$$

c) coinverses (antipodes)

$$S(M_i) = -M_i , \quad S(N_i) = -e^{\frac{P_0}{\kappa}} N_i + \epsilon_{ijk} e^{\frac{P_0}{2\kappa}} P_j M_k , \quad (16)$$

$$S(P_i) = -P_i , \quad S(P_0) = -P_0 . \quad (17)$$

The κ -Poincare algebra has two κ -deformed Casimirs describing mass and spin. The deformed bilinear mass Casimir looks as follows

$$C_\kappa^2(P_\mu) = C_\kappa^2(\vec{P}, P_0) = \left(2\kappa \sinh \left(\frac{P_0}{2\kappa} \right) \right)^2 - \vec{P}^2 . \quad (18)$$

Because in any basis $C_\kappa^2(P_\mu) = C_\kappa^2(S(P_\mu))$ we see from (17), (18) that any solution with $P_0 > 0$ has its "antiparticle counterpart" with $P_0 \rightarrow -P_0$.

The κ -deformed Minkowski space is defined as the translation sector of dual κ -Poincare group [15], [16] with algebraic relations (2) and the classical coproduct

$$\Delta(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu . \quad (19)$$

In order to introduce the noncommutative field operators one should define the κ -deformed noncommutative plane waves. There are possible various choices related with ordering ambiguity

of the time and space components of the plane waves (see e.g. [31]), as well as one can introduce the nonlinear transformations of the fourmomentum variables (see e.g. [29]). We shall choose the formula

$$:e^{ip_\mu \hat{x}^\mu}: = e^{\frac{i}{2}p_0 \hat{x}^0} e^{ip_i \hat{x}^i} e^{\frac{i}{2}p_0 \hat{x}^0}, \quad (20)$$

with simple Hermitean conjugation property

$$\left(:e^{ip_\mu \hat{x}^\mu}: \right)^\dagger = :e^{-ip_\mu \hat{x}^\mu}:. \quad (21)$$

From (2) and (20) follows the multiplication rule

$$:e^{ip_\mu \hat{x}^\mu}: :e^{iq_\mu \hat{x}^\mu}: = :e^{i\Delta_\mu(p,q) \hat{x}^\mu}:, \quad (22)$$

where the fourvector Δ_μ is determined by the coproduct (15)

$$\Delta_\mu(p, q) = \left(\Delta_0 = p_0 + q_0, \Delta_i = p_i e^{\frac{q_0}{2\kappa}} + q_i e^{-\frac{p_0}{2\kappa}} \right). \quad (23)$$

If we use the relation between two choices of noncommutative plane waves

$$:e^{ip_\mu \hat{x}^\mu}: = :e^{i\tilde{p}_\mu \hat{x}^\mu}: \equiv e^{i\tilde{p}_i \hat{x}^i} e^{i\tilde{p}_0 \hat{x}^0}, \quad (24)$$

where $\tilde{p}_\mu = (\tilde{p}_0 = p_0, \tilde{p}_i = e^{-\frac{p_0}{2\kappa}} p_i)$, the known formulae of the κ -deformed bicovariant differential calculus [30] provide

$$\hat{\partial}_i :e^{ip_\mu \hat{x}^\mu}: = e^{\frac{p_0}{2\kappa}} p_i :e^{ip_\mu \hat{x}^\mu}:, \quad (25)$$

and

$$\hat{\partial}_0 :e^{ip_\mu \hat{x}^\mu}: = \left[\kappa \left(e^{\frac{p_0}{2\kappa}} - 1 \right) + \frac{1}{2\kappa} C_\kappa^2(p_\mu) \right] :e^{ip_\mu \hat{x}^\mu}:. \quad (26)$$

Consequently, from (25), (26) follows that (see also [18], [30], [31])

$$\hat{\square} :e^{ip_\mu \hat{x}^\mu}: = C_\kappa^2(p_\mu) \left(1 - \frac{C_\kappa^2(p_\mu)}{4\kappa^2} \right) :e^{ip_\mu \hat{x}^\mu}:. \quad (27)$$

In the following section we shall apply the formulae (20)-(27) to the description of free quantum noncommutative fields.

3 Free κ -deformed quantum fields: new star product and microcausality.

We shall describe the κ -deformed quantum scalar free field on noncommutative Minkowski space (2) by the following κ -deformed Fourier transform

$$\hat{\phi}(\hat{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4 p A(p_0, \vec{p}) \delta(C_\kappa^2(\vec{p}, p_0) - M^2) :e^{ip_\mu \hat{x}^\mu}:. \quad (28)$$

As follows from the formula (27) the free fields satisfying noncommutative Klein-Gordon equation can be described as a superposition of two fields (28) with two κ -dependent masses - one physical, and second describing ghost field (see e.g. [18]).

We recall that in the quantum field (28) the noncommutative structures follows from two sources:

- i) The presence of noncommutative Minkowski space coordinates \hat{x}_μ (see (2)),
- ii) The noncommutativity of the Fourier modes $A(p_0, \vec{p})$.

If we solve the mass-shell condition $C_\kappa^2 - M^2 = 0$ one obtains

$$p_0^\pm = \pm \omega_\kappa(\vec{p}) \quad , \quad \omega_\kappa(\vec{p}) = 2\kappa \operatorname{arcsinh} \left(\frac{\sqrt{\vec{p}^2 + M^2}}{2\kappa} \right) . \quad (29)$$

Using the relation

$$\delta(C_\kappa^2(\vec{p}, p_0) - M^2) = \frac{1}{2\Omega_+(\vec{p})} \delta(p_0 - p_0^+) + \frac{1}{2\Omega_-(\vec{p})} \delta(p_0 - p_0^-) , \quad (30)$$

where³

$$2\Omega_\pm(\vec{p}) = \left| \frac{\partial}{\partial p_0} C_\kappa^2(\vec{p}, p_0) \right|_{p_0=p_0^\pm} = 2\kappa \sinh \left(\frac{\omega_\kappa(\vec{p})}{\kappa} \right) \equiv 2\Omega_\kappa(\vec{p}) , \quad (31)$$

and applying the decomposition into positive ($p_0 = \omega_\kappa(\vec{p})$) and negative ($p_0 = -\omega_\kappa(\vec{p})$) frequency parts

$$\hat{\phi}(\hat{x}) = \hat{\phi}_+(\hat{x}) + \hat{\phi}_-(\hat{x}) , \quad (32)$$

we get

$$\hat{\phi}_\pm(\hat{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{p}}{2\Omega_\kappa(\vec{p})} A(\pm\omega_\kappa(\vec{p}), \pm\vec{p}) :e^{\pm i p_\mu \hat{x}^\mu} :|_{p_0=\omega_\kappa(\vec{p})} . \quad (33)$$

In order to obtain the κ -deformed real scalar field (i.e. we identify the particles and antiparticles)

$$\left(\hat{\phi}_\pm(\hat{x}) \right)^\dagger = \hat{\phi}_\mp(\hat{x}) , \quad (34)$$

one should assume that

$$A^\dagger(\pm\omega_\kappa(\vec{p}), \pm\vec{p}) = A(\mp\omega_\kappa(\vec{p}), \mp\vec{p}) . \quad (35)$$

In standard quantum field theory, if $\kappa \rightarrow \infty$, the creation and annihilation operators ($\omega_\infty(\vec{p}) = \omega(\vec{p}) = \sqrt{\vec{p}^2 + M^2}$)

$$a(p) = A(\omega(\vec{p}), \vec{p}) \quad , \quad a^\dagger(p) = A(-\omega(\vec{p}), -\vec{p}) , \quad (36)$$

are quantized as follows

$$[a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0 \quad , \quad [a^\dagger(p), a(q)] = 2\omega(\vec{p}) \delta^{(3)}(\vec{p} - \vec{q}) . \quad (37)$$

The main question which now will be considered is how looks the κ -deformation of the relations (37) describing κ -deformed quantum free fields.

In order to represent the algebra of noncommutative fields on κ -Minkowski space (2) by the fields on classical space-time, we shall introduce the homomorphic \star -product multiplication by means of the Weyl map $:e^{ip_\mu \hat{x}^\mu} : \rightarrow e^{ip_\mu x^\mu}$ reproducing the relation (22), i.e.

$$:e^{ip_\mu \hat{x}^\mu} : \cdot :e^{iq_\mu \hat{x}^\mu} : \leftrightarrow e^{ip_\mu x^\mu} \star e^{iq_\mu x^\mu} = e^{i(p_0+q_0)x^0 + i\Delta_i(\vec{p}, \vec{q})x^i} . \quad (38)$$

³We choose the factor $\Omega_\kappa(\vec{p})$ having the standard $\kappa \rightarrow \infty$ limit equal to $\omega(\vec{p})$.

Now we shall introduce the new step - we assume that the star-product of the quantum free fields (28) affects also the mass-shell conditions. We propose the following new star-product \star_κ of two free κ -deformed fields

$$\hat{\phi}(\hat{x}) \cdot \hat{\phi}(\hat{x}) \leftrightarrow \varphi(x) \star_\kappa \varphi(x) = \frac{1}{(2\pi)^3} \int d^4p \int d^4q A(p_0, \vec{p}) A(q_0, \vec{q}) e^{i(p_0+q_0)x^0 + i\Delta_i(\vec{p}, \vec{q})x^i} \cdot \delta(C_\kappa^2(\vec{p}e^{\frac{q_0}{2\kappa}}, p_0) - M^2) \delta(C_\kappa^2(\vec{q}e^{-\frac{p_0}{2\kappa}}, q_0) - M^2). \quad (39)$$

In order to consider the bilocal product of noncommutative fields we extend the Weyl map (38) as follows (see also [22])

$$:e^{ip_\mu \hat{x}^\mu}: :e^{iq_\mu \hat{y}^\mu}: \leftrightarrow e^{ip_\mu x^\mu} \star e^{iq_\mu y^\mu} = e^{i(p_0 x^0 + q_0 y^0) + (p_i e^{\frac{q_0}{2\kappa}} x^i + q_i e^{-\frac{p_0}{2\kappa}} y^i)}. \quad (40)$$

From (40) we obtain by differentiation the braid relations between two copies of κ -Minkowski space

$$x_0 \star y_0 = x_0 y_0, \quad y_0 \star x_0 = y_0 x_0 \quad \Rightarrow \quad [x_0, y_0]_\star = 0, \quad (41)$$

$$x_i \star y_j = x_i y_j, \quad y_j \star x_i = y_j x_i \quad \Rightarrow \quad [x_i, y_j]_\star = 0, \quad (42)$$

$$x_0 \star y_i = \frac{i}{2\kappa} y_i + y_i x_0, \quad y_i \star x_0 = -\frac{i}{2\kappa} y_i + y_i x_0 \quad \Rightarrow \quad [x_0, y_i]_\star = \frac{i}{\kappa} y_i, \quad (43)$$

$$x_i \star y_0 = -\frac{i}{2\kappa} x_i + x_i y_0, \quad y_0 \star x_i = \frac{i}{2\kappa} x_i + x_i y_0 \quad \Rightarrow \quad [y_0, x_i]_\star = \frac{i}{\kappa} x_i. \quad (44)$$

For the bilocal product of free κ -deformed quantum fields we extend (39) in the following way

$$\hat{\phi}(\hat{x}) \cdot \hat{\phi}(\hat{y}) \leftrightarrow \varphi(x) \star_\kappa \varphi(y) = \frac{1}{(2\pi)^3} \int d^4p \int d^4q A(p_0, \vec{p}) A(q_0, \vec{q}) e^{i(p_0 x^0 + q_0 y^0) + (p_i e^{\frac{q_0}{2\kappa}} x^i + q_i e^{-\frac{p_0}{2\kappa}} y^i)} \cdot \delta(C_\kappa^2(\vec{p}e^{\frac{q_0}{2\kappa}}, p_0) - M^2) \delta(C_\kappa^2(\vec{q}e^{-\frac{p_0}{2\kappa}}, q_0) - M^2). \quad (45)$$

We stress that the star product \star_κ is different from the standard one, generated by the formula (40). It introduces new multiplication rule of two free fields, which is characterized by additional shifts $\vec{p} \rightarrow \vec{p}e^{\frac{q_0}{2\kappa}}$ and $\vec{q} \rightarrow \vec{q}e^{-\frac{p_0}{2\kappa}}$ of the threemomentum variables occuring in the arguments of the mass-shell deltas. The prescription (39) is defined only for the free fields, with fourmomenta restricted by κ -deformed mass-shell condition, and in interacting case can be used only for the description of κ -deformed perturbative expansions.

We shall relate the \star_κ -multiplication (45) with the κ -deformed statistics, described by the κ -deformation of the relations (37). Let us introduce new variables

$$\mathcal{P}_0 = p_0, \quad \mathcal{P}_i = p_i e^{\frac{q_0}{2\kappa}}, \quad \mathcal{Q}_0 = q_0, \quad \mathcal{Q}_i = q_i e^{-\frac{p_0}{2\kappa}}, \quad (46)$$

in the momentum integrals in (45). One gets

$$\varphi(x) \star_\kappa \varphi(y) = \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} A(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \vec{\mathcal{P}}) A(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \vec{\mathcal{Q}}) e^{i(\mathcal{P}_\mu x^\mu + \mathcal{Q}_\mu y^\mu)} \cdot \delta(C_\kappa^2(\vec{\mathcal{P}}, \mathcal{P}_0) - M^2) \delta(C_\kappa^2(\vec{\mathcal{Q}}, \mathcal{Q}_0) - M^2). \quad (47)$$

Using the formula (47) after the exchange $x \leftrightarrow y$ we have

$$[\varphi(x), \varphi(y)]_{\star_\kappa} = \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} \left[e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} A(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \vec{\mathcal{P}}) A(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \vec{\mathcal{Q}}) - e^{\frac{-3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} A(\mathcal{Q}_0, e^{-\frac{\mathcal{P}_0}{2\kappa}} \vec{\mathcal{Q}}) A(\mathcal{P}_0, e^{\frac{\mathcal{Q}_0}{2\kappa}} \vec{\mathcal{P}}) \right] \cdot e^{i(\mathcal{P}_\mu x^\mu + \mathcal{Q}_\mu y^\mu)} \delta(C_\kappa^2(\vec{\mathcal{P}}, \mathcal{P}_0) - M^2) \delta(C_\kappa^2(\vec{\mathcal{Q}}, \mathcal{Q}_0) - M^2). \quad (48)$$

Now, we introduce the κ -deformed creation and annihilation operators as follows

$$a_\kappa(p) = A(\omega_\kappa(\vec{p}), \vec{p}) \quad , \quad a_\kappa^\dagger(p) = A(-\omega_\kappa(\vec{p}), -\vec{p}) \quad . \quad (49)$$

We also define the following new κ -deformed multiplication for the operators (49)⁴

$$a_\kappa(p) \circ a_\kappa(q) = F_\kappa^{(2)}(p_0, q_0) a_\kappa\left(p_0, e^{-\frac{q_0}{2\kappa}} \vec{p}\right) a_\kappa\left(q_0, e^{\frac{p_0}{2\kappa}} \vec{q}\right) \quad , \quad (50)$$

$$a_\kappa^\dagger(p) \circ a_\kappa^\dagger(q) = F_\kappa^{(2)}(-p_0, -q_0) a_\kappa^\dagger\left(p_0, e^{\frac{q_0}{2\kappa}} \vec{p}\right) a_\kappa^\dagger\left(q_0, e^{-\frac{p_0}{2\kappa}} \vec{q}\right) \quad , \quad (51)$$

and

$$a_\kappa^\dagger(p) \circ a_\kappa(q) = F_\kappa^{(2)}(-p_0, q_0) a_\kappa^\dagger\left(p_0, e^{-\frac{q_0}{2\kappa}} \vec{p}\right) a_\kappa\left(q_0, e^{-\frac{p_0}{2\kappa}} \vec{q}\right) \quad , \quad (52)$$

$$a_\kappa(p) \circ a_\kappa^\dagger(q) = F_\kappa^{(2)}(p_0, -q_0) a_\kappa\left(p_0, e^{\frac{q_0}{2\kappa}} \vec{p}\right) a_\kappa^\dagger\left(q_0, e^{\frac{p_0}{2\kappa}} \vec{q}\right) \quad , \quad (53)$$

where $p_0 = \omega_\kappa(\vec{p})$, $q_0 = \omega_\kappa(\vec{q})$ and $F_\kappa^{(2)}(p_0, q_0) = e^{\frac{3}{2\kappa}(p_0 - q_0)}$ ⁵. Our basic postulate is the following set of κ -deformed creation and annihilation operators ($[A, B]_\circ := A \circ B - B \circ A$)

$$[a_\kappa(p), a_\kappa(q)]_\circ = [a_\kappa^\dagger(p), a_\kappa^\dagger(q)]_\circ = 0 \quad , \quad [a_\kappa^\dagger(p), a_\kappa(q)]_\circ = 2\Omega_\kappa(\vec{p})\delta^{(3)}(\vec{p} - \vec{q}) \quad , \quad (54)$$

where the κ -deformation of standard formulae (37) is contained in the deformed multiplication rules (50)-(53).

Let us introduce the \circ -product of two κ -deformed scalar fields as follows

$$\begin{aligned} \varphi(x) \circ \varphi(y) = & \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} A(\mathcal{P}_0, \vec{\mathcal{P}}) \circ A(\mathcal{Q}_0, \vec{\mathcal{Q}}) \cdot e^{i(\mathcal{P}x + \mathcal{Q}y)} \\ & \cdot \delta(C_\kappa^2(\vec{\mathcal{P}}, \mathcal{P}_0) - M^2) \delta(C_\kappa^2(\vec{\mathcal{Q}}, \mathcal{Q}_0) - M^2) \quad , \end{aligned} \quad (55)$$

⁴The formulae (50)-(53) differ from the ones studied in detail in [22] by the functional normalization factor $F_\kappa^{(2)}(\vec{p}, \vec{q})$; see however that (50) is a special case of the general formula (29) from [22].

⁵The derivation of the relations (50)-(53) is based on two steps. Firstly, we consider the consistency with non-Abelian addition law (15) for the threemomenta. Using the Hopf-algebraic formula

$$P_i \triangleright (a(p)a(q)) = (\Delta_{(1)} \triangleright a(p)) \cdot (\Delta_{(2)} \triangleright a(q)) \quad , \quad (*)$$

where we assume that (see also [11], [27])

$$P_i \triangleright a(p) = p_i a(p) \quad ,$$

we get

$$P_i \triangleright (a(p)a(q)) = p_i^{(1+2)} a(p)a(q) \quad ,$$

where

$$p_i^{(1+2)} = e^{\frac{q_0}{2\kappa}} p_i + e^{-\frac{p_0}{2\kappa}} q_i \quad .$$

We should modify threemomenta of the exchanged oscillators ($\vec{p} \rightarrow \vec{p}_\kappa, \vec{q} \rightarrow \vec{q}_\kappa$) in such a way that

$$P_i \triangleright (a(q_\kappa)a(p_\kappa)) = p_i^{(1+2)} a(q_\kappa)a(p_\kappa) \quad . \quad (**)$$

One gets, using (*), that

$$\vec{p}_\kappa = e^{\frac{q_0}{\kappa}} \vec{p} \quad , \quad \vec{q}_\kappa = e^{-\frac{p_0}{\kappa}} \vec{q} \quad .$$

In order to obtain classical threemomenta addition law we perform the second step: we change in the relations (*) and (**) the variables $(\vec{p}, \vec{q}) \rightarrow (e^{-\frac{q_0}{2\kappa}} \vec{p}, e^{\frac{p_0}{2\kappa}} \vec{q})$, what leads to the classical formula for the two-particle threemomenta. In such a way we obtain relation (50); the derivation of the relations (51)-(53) is analogues provided that

$$P_\mu \triangleright a^\dagger(p) = -p_\mu a^\dagger(p) \quad .$$

where the product $A(p_0, \vec{p}) \circ A(q_0, \vec{q})$ is determined via formula (49) by four definitions (50)-(53). If we introduce the formula inverse to the one given by (46), the nonlinear change of the momentum variables $(\mathcal{P}, \mathcal{Q}) \rightarrow (p, q)$ leads to the identity

$$\varphi(x) \star_{\kappa} \varphi(y) = \varphi(x) \circ \varphi(y) . \quad (56)$$

We would like to recall that the relation (56) for canonical $\theta_{\mu\nu}$ -deformation ($\theta_{\mu\nu} = \text{const}$) is known (see [11], eq. (3.8)). One can say that if we postulate the κ -statistics using the relation (54), we can derive the \star_{κ} -multiplication given by the formula (39) from the relation (56).

We see therefore that our new star product \star_{κ} can be equivalently described by the \circ -multiplication describing κ -deformation of the oscillator algebra. After simple calculation one can show that⁶

$$\begin{aligned} [\varphi(x), \varphi(y)]_{\star_{\kappa}} &= [\varphi(x), \varphi(y)]_{\circ} = \frac{-i}{(2\pi)^3} \int \frac{d^3 p}{\Omega_{\kappa}(\vec{p})} \sin(\omega_{\kappa}(\vec{p})(x_0 - y_0)) e^{i\vec{p}(\vec{x} - \vec{y})} = \\ &= \frac{1}{i} \Delta_{\kappa}(x - y; M^2) . \end{aligned} \quad (57)$$

It should be noted that the commutator (57) was firstly obtained in [25] by using "naive" deformation of the free quantum K-G fields (the "ad hoc" insertion of κ -deformed mass-shell condition).

Let us consider the equal time properties of the κ -deformed Pauli-Jordan commutator function. It follows from (57) that

$$\Delta_{\kappa}(x - y; M^2)|_{x_0=y_0} = 0 \quad \Leftrightarrow \quad [\varphi(x_0, \vec{x}), \varphi(x_0, \vec{y})]_{\star_{\kappa}} = 0 , \quad (58)$$

$$\partial_0 \Delta_{\kappa}(x - y; M^2)|_{x_0=y_0} = [\partial_0 \varphi(x_0, \vec{x}), \varphi(x_0, \vec{y})]_{\star_{\kappa}} = \delta_{\kappa}^{(3)}(\vec{x} - \vec{y}) , \quad (59)$$

where

$$\delta_{\kappa}^{(3)}(\vec{x} - \vec{y}) = \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{\omega_{\kappa}(\vec{p}^2)}{\Omega_{\kappa}(\vec{p}^2)} e^{i\vec{p}(\vec{x} - \vec{y})} \xrightarrow{\kappa \rightarrow \infty} \delta^{(3)}(\vec{x} - \vec{y}) , \quad (60)$$

is the κ -deformed nonlocal counterpart of standard Dirac delta function. It appears however that if we define "quantum" time derivative

$$\partial_0^{\kappa} \equiv i\kappa \sinh\left(\frac{1}{i} \frac{\partial_0}{\kappa}\right) = \kappa \sin\left(\frac{\partial_0}{\kappa}\right) \xrightarrow{\kappa \rightarrow \infty} \partial_0 , \quad (61)$$

one gets the local expression for any value of κ

$$\partial_0^{\kappa} \Delta_{\kappa}(x - y; M^2)|_{x_0=y_0} = \delta^{(3)}(\vec{x} - \vec{y}) . \quad (62)$$

The formulae (59)-(62) describe the equal time relations representing the κ -causality of κ -deformed quantum fields. It should be stressed that such results require the κ -deformed algebra (50)-(53) of the field oscillators.

4 The algebra of κ -deformed bosonic creation and annihilation operators.

In order to study the full algebra of creation and annihilation operators one should extend the relations (50)-(53) to the \circ -product of arbitrary polynomials of creation and annihilation operators.

⁶See also the relations (4), (5).

Because due to the relation (49) one can describe the relations (51)-(53) by suitable extension of the relation (50) to negative energies, further we shall consider only the products of the creation operators $a_\kappa(p)$ with arbitrary values of p (sign of p_0 is not specified).

The formula (50) can be extended to an arbitrary product of n κ -deformed creation oscillators in the following way

$$a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) = F_\kappa^{(n)}(p_0^{(1)}, \dots, p_0^{(n)}) a_\kappa(p_0^{(1)}, \chi_n^{(1)}(p_0^{(1)}, \dots, p_0^{(n)}) \vec{p}^{(1)}) \dots \\ \dots a_\kappa(p_0^{(k)}, \chi_n^{(k)}(p_0^{(1)}, \dots, p_0^{(n)}) \vec{p}^{(k)}) \dots a_\kappa(p_0^{(n)}, \chi_n^{(n)}(p_0^{(1)}, \dots, p_0^{(n)}) \vec{p}^{(n)}) , \quad (63)$$

where

$$\chi_n^{(k)}(p_0^{(1)}, \dots, p_0^{(n)}) = \exp \frac{1}{2\kappa} \left(\sum_{j=1}^{k-1} p_0^{(j)} - \sum_{j=k+1}^n p_0^{(j)} \right) , \quad (64)$$

$$F_\kappa^{(n)}(p_0^{(1)}, \dots, p_0^{(n)}) = \left[\prod_{k=1}^n \chi_n^{(k)}(p_0^{(1)}, \dots, p_0^{(n)}) \right]^3 = \exp \left(\frac{3}{2\kappa} \sum_{k=1}^n (n+1-2k) p_0^{(k)} \right) ,$$

or using more compact notation

$$a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) = F_\kappa^{(n)}(p_0^{(1)}, \dots, p_0^{(n)}) a_\kappa(p_0^{(1)}, \vec{\mathcal{P}}_n^{(1)}) \dots a_\kappa(p_0^{(n)}, \vec{\mathcal{P}}_n^{(n)}) , \quad (65)$$

with

$$\vec{\mathcal{P}}_n^{(k)} = \exp \frac{1}{2\kappa} \left(\sum_{j=1}^{k-1} p_0^{(j)} - \sum_{j=k+1}^n p_0^{(j)} \right) \vec{p}^{(k)} = \chi_n^{(k)}(p_0^{(1)}, \dots, p_0^{(n)}) \vec{p}^{(k)} . \quad (66)$$

Let us observe that the factors $\chi_n^{(k)}$ are determined by the n -th iteration of the coproduct of the fourmomenta in the basis with the basic coproducts (14), (15)

$$\Delta^{(n)}(P_0) = \sum_{k=1}^n \underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes P_0 \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-k} , \quad (67)$$

$$\Delta^{(n)}(P_i) = \sum_{k=1}^n \underbrace{e^{\frac{P_0}{2\kappa}} \otimes \dots \otimes e^{\frac{P_0}{2\kappa}}}_{k-1} \otimes P_i \otimes \underbrace{e^{-\frac{P_0}{2\kappa}} \otimes \dots \otimes e^{-\frac{P_0}{2\kappa}}}_{n-k} . \quad (68)$$

The relation (68) encodes the deformed addition law for the threemomenta, but if we assume that (see also [11], [27])

$$P_\mu \triangleright a_\kappa(p) = p_\mu a_\kappa(p) , \quad (69)$$

we have (we use Sweedler notation for n -fold coproduct $\Delta^{(n)} = \Delta_{(1)}^{(n)} \otimes \dots \otimes \Delta_{(n)}^{(n)}$)

$$P_0 \triangleright (a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)})) = F_\kappa^{(n)} \omega \left(\Delta^{(n)}(P_0) \triangleright (a_\kappa(\mathcal{P}_n^{(1)}) \otimes \dots \otimes a_\kappa(\mathcal{P}_n^{(n)})) \right) = \\ = F_\kappa^{(n)} \omega \left(\Delta_{(1)}^{(n)}(P_0) \triangleright a_\kappa(\mathcal{P}_n^{(1)}) \otimes \dots \otimes \Delta_{(n)}^{(n)}(P_0) \triangleright a_\kappa(\mathcal{P}_n^{(n)}) \right) = \\ = \sum_{j=1}^n p_0^{(j)} a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) , \quad (70)$$

where $\omega(A \otimes B) = AB$, and

$$\begin{aligned}
\vec{P} \triangleright (a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)})) &= F_\kappa^{(n)} \omega \left(\Delta^{(n)}(\vec{P}) \triangleright (a_\kappa(\mathcal{P}_n^{(1)}) \otimes \dots \otimes a_\kappa(\mathcal{P}_n^{(n)})) \right) = \\
&= F_\kappa^{(n)} \omega \left(\Delta_{(1)}^{(n)}(\vec{P}) \triangleright a_\kappa(\mathcal{P}_n^{(1)}) \otimes \dots \otimes \Delta_{(n)}^{(n)}(\vec{P}) \triangleright a_\kappa(\mathcal{P}_n^{(n)}) \right) = \\
&= \sum_{j=1}^n \vec{\mathcal{P}}_n^{(j)} \left(\chi_n^{(j)}(p_0^{(1)}, \dots, p_0^{(n)}) \right)^{-1} a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) = \\
&= \sum_{j=1}^n \vec{p}^{(j)} a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) .
\end{aligned} \tag{71}$$

We see therefore that the modification of the momentum arguments in the κ -deformed oscillators (50)-(53) exactly cancels non-Abelian factors in the quantum addition law, governed by the coproduct (68).

Using the relations (67), (68) one can also explain the meaning of our new star-product binary multiplication (45). Let us introduce new set of threemomentum variables (compare with (66))

$$\hat{\vec{\mathcal{P}}}_n^{(i)} = \chi_n^i(p_0^{(1)}, \dots, p_0^{(n)})^{-1} \vec{p}^{(i)} , \tag{72}$$

which describe the i -th particle contribution to the total n -particle threemomentum, obtained from the coproduct (68). For $n = 2$ one can interpret the threevectors $\hat{\vec{\mathcal{P}}}_2^{(1)}, \hat{\vec{\mathcal{P}}}_2^{(2)}$ as describing the first and second particle threemomenta in a two-particle state $a_\kappa(p_1)a_\kappa(p_2)|0\rangle$, because

$$\vec{P} \triangleright (a_\kappa(p^{(1)})a_\kappa(p^{(2)})) = \omega \left(\Delta^{(2)} \triangleright (a_\kappa(p^{(1)}) \otimes a_\kappa(p^{(2)})) \right) = \left(\hat{\vec{\mathcal{P}}}_2^{(1)} + \hat{\vec{\mathcal{P}}}_2^{(2)} \right) (a_\kappa(p^{(1)})a_\kappa(p^{(2)})) . \tag{73}$$

The deformation of the product of two mass-shell conditions in the formula (45) can be described by the following replacement ($i = 1, 2$)

$$C_\kappa^2(p_0^{(i)}, \vec{p}^{(i)}) \rightarrow C_\kappa^2(p_0^{(i)}, \hat{\vec{\mathcal{P}}}_2^{(i)}) , \tag{74}$$

i.e. we put the threemomenta $\hat{\vec{\mathcal{P}}}_2^{(1)}, \hat{\vec{\mathcal{P}}}_2^{(2)}$ on κ -deformed mass-shell.

The relation (74) can be extended to the product of n κ -deformed free fields containing product of n κ -deformed mass-shell conditions. In a such case the relation (56) we extend as follows

$$\varphi(x_1) \star_\kappa \dots \star_\kappa \varphi(x_n) = \varphi(x_1) \circ \dots \circ \varphi(x_n) , \tag{75}$$

where the rhs of (75) is defined with the use of the relation (63). In order to define consistently the lhs of (75) we should replace in the i -th mass-shell $\vec{p}^{(i)} \rightarrow \hat{\vec{\mathcal{P}}}_n^{(i)}$ (see (72)).

The κ -deformed algebra of oscillators can be formulated in a way consistent with the associativity property of the product (63). Indeed, one can define the product $(a_\kappa(p^{(1)}) \cdot \dots \cdot a_\kappa(p^{(n)})) \circ (a_\kappa(q^{(1)}) \cdot \dots \cdot a_\kappa(q^{(m)}))$ in such a way that for any n, m the relation

$$\begin{aligned}
(a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)})) \circ (a_\kappa(q^{(1)}) \circ \dots \circ a_\kappa(q^{(m)})) &= \\
&= (a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}) \circ a_\kappa(q^{(1)}) \circ \dots \circ a_\kappa(q^{(m)})) ,
\end{aligned} \tag{76}$$

is valid. If we insert the relation (76) into the relation (75) we see that the star multiplication \star_κ of κ -deformed free fields is also associative.

5 κ -deformed Fock space, fourmomentum conservation laws and statistics.

Let us introduce the normalized vacuum state in standard way

$$a_\kappa^\dagger(p_0, \vec{p})|0\rangle = 0 \quad , \quad \langle 0|0\rangle = 1 \quad , \quad (77)$$

where $p_0 = \omega_\kappa(\vec{p})$. The one-particle state is defined as follows⁷

$$|\vec{p}\rangle = a_\kappa(p_0, \vec{p})|0\rangle \quad , \quad (78)$$

and from (69) we get

$$P_\mu |\vec{p}\rangle = p_\mu |\vec{p}\rangle \quad . \quad (79)$$

We define two-particle states in the following way

$$|\vec{p}, \vec{q}\rangle = a_\kappa(p_0, \vec{p}) \circ a_\kappa(q_0, \vec{q})|0\rangle \quad . \quad (80)$$

By using (67) and (68) we obtain⁸

$$P_\mu |\vec{p}, \vec{q}\rangle = (p_\mu + q_\mu) |\vec{p}, \vec{q}\rangle \quad . \quad (81)$$

From (37) follows the standard bosonic symmetry

$$|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle \quad . \quad (82)$$

Taking into account associativity property of κ -multiplication (see (76)) one can define the n-particle state by the κ -deformed product of n oscillators. We obtain

$$|\vec{p}^{(1)}, \dots, \vec{p}^{(k)}, \dots, \vec{p}^{(n)}\rangle = a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(k)}) \circ \dots \circ a_\kappa(p^{(n)})|0\rangle \quad . \quad (83)$$

Using the coassociative coproduct (67) and (68) of n-th order, we can calculate from (70), (71) the total momentum of the state (83), which appears to be given by classical formula

$$\begin{aligned} P_\mu |\vec{p}^{(1)}, \dots, \vec{p}^{(k)}, \dots, \vec{p}^{(n)}\rangle &= [\Delta^{(n)}(P_\mu) \triangleright (a_\kappa(p^{(1)}) \circ \dots \circ a_\kappa(p^{(n)}))] |0\rangle = \\ &= \sum_{i=1}^n p_\mu^{(i)} |\vec{p}^{(1)}, \dots, \vec{p}^{(k)}, \dots, \vec{p}^{(n)}\rangle \quad . \end{aligned} \quad (84)$$

We also get from (37) the known bosonic symmetry property

$$|\vec{p}^{(1)}, \dots, \vec{p}^{(i)}, \dots, \vec{p}^{(j)}, \dots, \vec{p}^{(n)}\rangle = |\vec{p}^{(1)}, \dots, \vec{p}^{(j)}, \dots, \vec{p}^{(i)}, \dots, \vec{p}^{(n)}\rangle \quad . \quad (85)$$

In order to complete the structure of κ -deformed Fock space we should define dual vectors and scalar product. We define the dual space in analogy to the relations (83)

$$\langle \vec{k}^{(1)}, \dots, \vec{k}^{(n)} | = \langle 0 | a_\kappa^\dagger(k_0^{(1)}, \vec{k}^{(1)}) \circ \dots \circ a_\kappa^\dagger(k_0^{(n)}, \vec{k}^{(n)}) \quad . \quad (86)$$

⁷If we consider single oscillators and one-particle states one can put $a_\kappa(p_0, \vec{p}) \equiv a(p_0, \vec{p})$, because the κ -deformation appears only if we multiply the oscillators.

⁸We should stress that two quanta in (80) are forming an intertwined system and should not be consider as a superposition of two independent modes. The intertwining effect puts each component of 2-particle state off-shell. This footnote contains the answer to the criticism presented in [27].

We define κ -deformed scalar product of the basic vectors (83) and (86) as follows

$$\begin{aligned} < \vec{k}^{(1)}, \dots, \vec{k}^{(m)} | \vec{p}^{(1)}, \dots, \vec{p}^{(n)} >_{\kappa} := < \vec{k}^{(1)}, \dots, \vec{k}^{(m)} | \circ | \vec{p}^{(1)}, \dots, \vec{p}^{(n)} > = \\ &= < 0 | a_{\kappa}^{\dagger}(k_0^{(1)}, \vec{k}^{(1)}) \circ \dots \circ a_{\kappa}^{\dagger}(k_0^{(m)}, \vec{k}^{(m)}) \circ a_{\kappa}(p_0^{(1)}, \vec{p}^{(1)}) \circ \dots \circ a_{\kappa}(p_0^{(n)}, \vec{p}^{(n)}) | 0 >, \end{aligned} \quad (87)$$

and, using (37), it is easy to show that

$$\begin{aligned} < \vec{k}^{(1)}, \dots, \vec{k}^{(m)} | \vec{p}^{(1)}, \dots, \vec{p}^{(n)} >_{\kappa} = \\ &= \delta_{nm} \sum_{\text{perm}(i_1, \dots, i_n)} \delta^{(3)}(\vec{p}^{(1)} - \vec{k}^{(i_1)}) \dots \delta^{(3)}(\vec{p}^{(n)} - \vec{k}^{(i_n)}). \end{aligned} \quad (88)$$

We see therefore that the fourmomentum eigenvalues and scalar products in κ -deformed Fock space are identical with the ones characterizing standard bosonic Fock space. We see also from (85) that the statistics of n-particle state remains bosonic⁹.

It appears that the notion of κ -multiplication is very useful. It describes the κ -deformation of the oscillators algebra as well as the operator-valued metric defining scalar product in κ -Fock space.

6 κ -deformed interaction vertex in $\lambda\varphi^4$ theory.

In order to define the perturbative vertex in noncommutative $\lambda\varphi^4$ theory we shall integrate over space-time the following product of κ -deformed free fields

$$\begin{aligned} \varphi(x) \star_{\kappa} \varphi(x) \star_{\kappa} \varphi(x) \star_{\kappa} \varphi(x) &= \int \prod_{k=1}^4 d^4 p^{(k)} \delta \left(C_{\kappa}^2(\vec{p}^{(k)}, p_0^{(k)}) - M^2 \right) A(p_0^{(k)}, \vec{p}^{(k)}) \cdot \\ &\quad \cdot e^{ip_{\mu}^{(1)} x^{\mu}} \star \dots \star e^{ip_{\mu}^{(4)} x^{\mu}} = \\ &= \int \prod_{k=1}^4 d^4 p^{(k)} \delta \left(C_{\kappa}^2(\vec{p}^{(k)}, p_0^{(k)}) - M^2 \right) A(p_0^{(1)}, \vec{p}^{(1)}) \circ \dots \circ A(p_0^{(4)}, \vec{p}^{(4)}) \cdot \\ &\quad \cdot \exp \left(i \sum_{k=1}^4 p_{\mu}^{(k)} x^{\mu} \right). \end{aligned} \quad (89)$$

Performing the integration, one gets ($P_{\mu} = \sum_{k=1}^4 p_{\mu}^{(k)}$)

$$\begin{aligned} \int d^4 x \varphi(x) \star_{\kappa} \varphi(x) \star_{\kappa} \varphi(x) \star_{\kappa} \varphi(x) &= \int d^4 x \int \prod_{k=1}^4 d^4 p^{(k)} \delta \left(C_{\kappa}^2(\vec{p}^{(k)}, p_0^{(k)}) - M^2 \right) e^{iP_{\mu} x^{\mu}} \cdot \\ &\quad \cdot A(p_0^{(1)}, \vec{p}^{(1)}) \circ \dots \circ A(p_0^{(4)}, \vec{p}^{(4)}) = \\ &= \int \prod_{k=1}^4 d^4 p^{(k)} \delta \left(C_{\kappa}^2(\vec{p}^{(k)}, p_0^{(k)}) - M^2 \right) \delta^{(4)}(P_{\mu}) \cdot A(p_0^{(1)}, \vec{p}^{(1)}) \circ \dots \circ A(p_0^{(4)}, \vec{p}^{(4)}) \cdot \end{aligned} \quad (90)$$

From (90) we see that the fourmomentum Dirac delta is classical and describes Abelian conservation law of fourmomenta at the vertex.

⁹We define the statistics as define by the symmetry properties of n-particle states.

The momentum space formula (90) is the proper input describing κ -deformed Feynman diagrams in $\lambda\varphi^4$ theory. Because the κ -deformed mass-shell (18) due to the formula [32]¹⁰

$$\frac{\sinh x}{x} = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{ix}{n\pi}\right) \left(1 - \frac{ix}{n\pi}\right), \quad (91)$$

contains infinite number of complex-conjugated poles (see e.g. [18], [33]), the problem of defining the κ -deformed propagator requires special care, in particular, the appropriate notion of time ordering. Indeed, let us observe that (see e.g. [34])

$$2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) f(t) = \frac{f(t + \Delta t) - f(t - \Delta t)}{\Delta t} \quad ; \quad \Delta t = \frac{i}{2\kappa}, \quad (92)$$

if $P_0 = \frac{1}{i}\partial_t$. We see that in κ -deformed field theory the continuous time derivatives are replaced by finite difference equations and the κ -deformed Green functions (see e.g. (62) for equal time limit) satisfy also the finite time difference equations.

The problem of space-time picture of κ -causality and proper definition of κ -deformed Feynman propagator is now under consideration.

7 Outlook.

Let us recall our main result. We have introduced new κ -deformation of creation and annihilation operators, which inserted in the free quantized fields provide deformed fields with c-number κ -deformed Pauli-Jordan commutator function (4). It appears that κ -deformation of free fields is equivalently described by a new star product \star_κ (see (39) and (45)). The κ -deformed oscillators were used for the construction of κ -deformed Fock space with multiparticle states, described by momenta which are added by using the Abelian addition law.

We would like to point out that due to the modification of κ -deformed mass-shell deltas (see (45)) the κ -deformed oscillators entering the algebra in Sect. 4 are beyond the standard κ -deformed mass-shell. Our modification of mass-shell condition is however essential in the derivation of c-number field commutator (57). We recall that recently (see [27], [35], [36]) there were proposed analogous forms of binary algebras for κ -deformed oscillators, where similarly as in our case, there is assumed the modification of three-momentum dependence under the exchange of two creation (annihilation) operators. It should be stressed however that in [27], [35] the fourmomenta describing the arguments of all κ -deformed oscillators in binary algebraic relations are put on-shell. It can be argued for more general κ -deformed statistics [37] that specific modification of mass-shell condition for the κ -oscillators in binary relations is necessary for the derivation of the c-number commutator for the free κ -deformed fields.

Our aim is to obtain the perturbative framework for the κ -deformed local field theory. We conjecture that the Feynman rules differs from the standard field framework only by different choice of the propagators, which satisfy the inhomogeneous κ -deformed Klein-Gordon equation. For this purpose one has to derive in the commutative framework with \star_κ multiplication the Gell-Mann-Low expansion for κ -deformed Green functions and the κ -deformed Dyson formula. For the applications it is important to extend the framework to fermionic Dirac and vectorial gauge fields. It should be recalled that the κ -deformed free Dirac and Maxwell fields were already considered before [38]. These problems are now under consideration.

¹⁰We apply the formula (91) by putting $x = \frac{p_0}{2\kappa}$.

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